Exercises for 'Topics in complex analysis'

(26/11/2025)

H 11.1 (True or false?)
Prove the following statements or give a counterexample.

(i) The image of a simply connected domain under a non-constant holomorphic function is again a simply connected domain.

(ii) The image of a simply connected domain under an injective holomorphic function is again a simply connected domain.

(iii) The complex plane is not biholomorphically equivalent to any simply connected domain $G \subsetneq \mathbb{C}$.

(iv) The set $\mathbb{C} \setminus B_r(z_0)$ with $z_0 \in \mathbb{C}$ and $r > 0$ is simply connected.

H 11.2 (Schwarz lemma on simply connected domains)
Let $G \subsetneq \mathbb{C}$ be a simply connected domain. Given $a \in G$ we denote by $\mathrm{Hol}_a(G)$ the set of holomorphic functions $f : G \to G$ such that $f(a) = a$. Prove that $|f'(a)| \leq 1$ for all $f \in \mathrm{Hol}_a(G)$. Moreover, show that $f \in \mathrm{Hol}_a(G)$ is bijective if and only if $|f'(a)| = 1$.
Hint: Use the function given by the Riemann mapping theorem.

H 11.3 (Singularities of injective holomorphic maps)
Let $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and injective. Prove that either z_0 is a removable singularity and the continuous extension to z_0 is still injective or z_0 is a pole of first order.
Hint: Rule out an essential singularity using Picard's great theorem.

H 11.4 (Rigidity of biholomorphic maps)
a) Let $f : B_1(0) \to B_1(0)$ be a biholomorphic map such that $f(a) = a$ and $f(b) = b$ for two distinct $a, b \in B_1(0)$. Prove that $f(z) = z$ for all $z \in B_1(0)$.

b) Show that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and injective if and only if $f(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$.
Hint: Try to apply Exercise H 11.3.

c) Show that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is holomorphic and injective if and only if $f(z) = az$ or $f(z) = az^{-1}$ for some $a \in \mathbb{C} \setminus \{0\}$.

d) Let $G \subsetneq \mathbb{C}$ be a simply connected domain and let $f \in \mathrm{Hol}_a(G)$ (cf. Exercise H 11.2) be biholomorphic. Show that $f'(a) \in (0, +\infty)$ implies that $f(z) = z$ for all $z \in G$.

H 11.5 (Examples of the Riemann mapping theorem)
In this exercise we provide biholomorphic maps $f : G \to B_1(0)$ for some special sets G.

a) Show that the map $z \mapsto \frac{z-i}{z+i}$ is biholomorphic from $\mathbb{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$ to $B_1(0)$.

b) Find a biholomorphic map $f : \mathbb{C} \setminus (-\infty, 0] \to B_1(0)$.